Non-Harmonic Analysis of Weighted Pseudo-differential Operators

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Outline

- Motivation
- Overview of Non-harmonic Analysis
- Global Symbolic Calculus
- Weighted Elliptic Operators and related results
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Motivation

- On $\mathbb{R}^n$, the Hörmander symbol class, $S^m_{\rho,\delta}$, $m \in \mathbb{R}$, $0 \le \rho, \delta \le 1$,

$$|(\partial_\xi^\alpha \partial_x^\beta \sigma)(x,\xi)| \le C_{\alpha,\beta}(1+|\xi|)^{m-\rho|\alpha|+\delta|\beta|}$$

- Using Mikjlin Hörmander Multiplier theorem it can be shown that pseudo-differential operators associated with $S^0_{1,0}$, is L^p-bounded. But for p$\neq 2$, *these operators with symbols in* $\mathrm{S}^0_{\rho,0}$, $0 < \rho < 1$, are not L^p-bounded.
- Taylor introduced a new subclass, $M^m_{\rho,0}$, of $S^0_{\rho,0}$ to overcome this problem.
- Garello and Morando defined a weighted version of Taylor's one by replacing $\sqrt{1+|\xi|^2}$ by a more general positive weight function $\Lambda(\xi)$.

Weight Function

- Weight Function: $\Lambda \in C^\infty(\mathbb{R}^n)$, positive function,

  i. $$C_0(1+|\xi|)^{\mu_0} \leq \Lambda(\xi) \leq C_1(1+|\xi|)^{\mu_1},$$

  $\xi \in \mathbb{R}^n$, μ_0, μ_1, C_0 and C_1 are constants with $\mu_0 \leq \mu_1$ and $C_0 \leq C_1$.

  ii. for all multi-indices $\alpha, \gamma \in \mathbb{N}_0^n$ with $\gamma_j \in \{0,1\}, j = 0, 1, 2, \dots, n$ there exist a positive constant $C_{\alpha,\gamma}$ such that

  $$|\xi^\gamma \partial_\xi^{\alpha+\gamma}\Lambda)(\xi) \leq C_{\alpha,\gamma}\Lambda(\xi)^{1-\frac{1}{\mu}|\alpha|},$$

  $\mu \geq \mu_1$, $x, \xi \in \mathbb{R}^n$.

Example

For $n = 2$, $\Lambda(\xi) = \sqrt{1+\xi_1^6+\xi_1^4\xi_2^4+\xi_2^6}$ satisfies with $\mu_0 = 3$, $\mu_1 = 4$ and $\mu = 6$.

Weighted Symbol Class

Let $m \in \mathbb{R}$ and $\rho \in (0, \frac{1}{\mu}]$, $\mu \geq \mu_1$

- $S^m_{\rho,\Lambda}$: $\sigma \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that

$$|(\partial_x^\alpha \partial_\xi^\beta \sigma)(x,\xi)| \leq C_{\alpha,\beta} \Lambda(\xi)^{m-\rho|\beta|},$$

  for all multi-indices α, β, $C_{\alpha,\beta} > 0$, constant, $x, \xi \in \mathbb{R}^n$.

- $M^m_{\rho,\Lambda}$: $\xi^\gamma(\partial_\xi^\gamma \sigma)(x,\xi) \in S^m_{\rho,\Lambda}$, for all multi-indices γ with $\gamma_j \in \{0,1\}$, $j = 1, 2, \ldots, n$.

Weighted Pseudo-differential Operators:

$$(T_\sigma \phi)(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \sigma(x,\xi) \widehat{\phi}(\xi) d\xi, \quad x \in \mathbb{R}^n, \ \phi \in \mathcal{S}(\mathbb{R}^n),$$

where

$$\widehat{\phi}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} \phi(x) dx, \qquad \xi \in \mathbb{R}^n.$$

A Short Overview

- $T_\sigma : \mathcal{S} \to \mathcal{S}$ is a continuous linear mapping.
- Symbolic calculus has been developed earlier. [Garello + Morando (2005); Wong (2006)]

For $\sigma \in M^m_{\rho,\Lambda}$, $u \in \mathcal{S}'$, $T_\sigma u : \mathcal{S} \to \mathbb{C}$ is defined by $(T_\sigma u)(\phi) = u(\overline{T^*_\sigma \overline{\phi}})$.

- $T_\sigma : \mathcal{S}' \to \mathcal{S}'$ is a continuous linear mapping.
- M-elliptic: For $\sigma \in M^m_{\rho,\Lambda}$, $m \in \mathbb{R}$, $\exists\ C, R > 0$ such that
$$|\sigma(x,\xi)| \geq C\Lambda^m(\xi), \quad |\xi| \geq R.$$
- Parametrix: For $\sigma \in M^m_{\rho,\Lambda}$, M-elliptic, $\exists\ \tau \in M^{-m}_{\rho,\Lambda}$ such that
$$T_\sigma T_\tau = I + R$$
and
$$T_\tau T_\sigma = I + S,$$
where R, S are pseudo-differential operators with symbols in $\cap_{k\in\mathbb{R}} M^k_{\rho,\Lambda}$.

Weight Function on $\mathbb{Z}$

- Weight Function:

1. Λ is a weight function if there exist suitable $\mu_0, \mu_1 > 0$, $\mu_0 \leq \mu_1$ and $C_0, C_1 > 0$ such that

$$C_0(1+|k|)^{\mu_0} \leq \Lambda(k) \leq C_1(1+|k|)^{\mu_1},$$

$k \in \mathbb{Z}$.

2. There exists a real constant μ such that $\mu \geq \mu_1$ and for all $\alpha, \gamma \in \mathbb{N}_0$ with $\gamma_j \in \{0,1\}$, $j = 1, 2, ..., n$, we can find a positive constant $C_{\alpha,\gamma}$ such that

$$\left|k^{\gamma} \Delta_k^{\alpha+\gamma} \Lambda(k)\right| \leq C_{\alpha,\gamma} \Lambda(k)^{1-\frac{1}{\mu}\alpha}, \quad k \in \mathbb{Z}.$$

Weighted Kohn-Nirenberg Symbol Class

Let $m \in \mathbb{R}$ and $\rho \in \left(0, \frac{1}{\mu}\right]$.

- Kohn-Nirenberg Symbol Class:
  $S^m_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$: Set of all functions $\sigma : \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ which are smooth in x, $\forall k \in \mathbb{Z}$ and for all $\alpha, \beta \in \mathbb{N}_0$ with $\gamma \in \{0, 1\}$, there is a constant $C_{\alpha,\gamma} > 0$ such that

$$\left|\Delta_k^\alpha \partial_x^\beta \sigma(x, k)\right| \leq C_{\alpha,\beta} \Lambda(k)^{m-\rho\alpha}.$$

- $M^m_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$: $\sigma : \mathbb{T} \times \mathbb{Z}$ such that,

$$k^\gamma \Delta_k^\gamma \sigma(x, k) \in S^m_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z}).$$

- Pseudo-differential operator, T_σ, is defined as

$$T_\sigma f(x) = \sum_{k \in \mathbb{Z}} e^{2\pi i x \cdot k} \sigma(x, k) \widehat{f}(k),$$

where $f \in C^\infty(\mathbb{T})$.

Boundedness

Theorem

Let $\sigma \in M^0_{\rho,\Lambda}(\mathbb{T}\times\mathbb{Z})$, $-\infty < m < \infty$. *Then* $T_\sigma : L^p(\mathbb{T}) \to L^p(\mathbb{T})$ *is a bounded linear operator for* $1 < p < \infty$.

- Bessel potential, J_s : the Ψ-DO with symbol σ_s given by

$$\sigma_s(k) = (\Lambda(k))^{-s}, \quad k \in \mathbb{Z}.$$

- Sobolev Space, $H^{s,p}_\Lambda = \{u \in \mathcal{D}'(\mathbb{T}) : J_{-s}u \in L^p(\mathbb{T})\}$. $H^{s,p}$ is a Banach space with norm $||.||_{s,p}$ given by

$$||u||_{s,p,\Lambda} = ||J_{-s}u||_{L^p(\mathbb{T})}.$$

Theorem

Let $\sigma \in M^m_{\rho,\Lambda}(\mathbb{T}\times\mathbb{Z})$, $-\infty < m < \infty$. *Then* $T_\sigma : H^{s,p} \to H^{s-m,p}$ *is a bounded linear operator for* $1 < p < \infty$.

Overview of global (harmonic) quantization theory

- Analysis on compact Lie groups. R.+Turunen, Pseudo-differential operators and symmetries, Birkhäuser, 2010
  With further developments: Turunen, Wirth, Dasgupta, Garetto, Tikonov, Cardona, Kumar, and Kirillov among many others.
- Analysis on nilpotent Lie groups. Fischer+R., Quantization on nilpotent Lie groups, Birkhäuser, Progress in Math., 2016.
- Analysis on locally compact type 1 groups.Mantoiu+R., Pseudo-differential operators, Wigner transform and Weyl systems on type 1 locally compact groups, Doc. Math. 2017.
- Analysis on the lattice $\mathbb{Z}^n$. Botchway+Kibiti+R., JFA 2020.
- Global quantization on compact manifolds. R+Delgado, J. d'Analyse Math, 2018.
- Global analysis on locally compact groups, quantum groups. JFA 2020, +Majid CMP 2018.

Harmonic Analysis of Ψ-DOs

Pseudo-differential operators on $\mathbb{R}^n$ [Kohn-Nirenberg, Hörmander, 1965]:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \quad T_\sigma f(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} \sigma(x,\xi) \widehat{f}(\xi) d\xi,$$

$$\left| \partial_\xi^\alpha \partial_x^\beta \sigma(x,\xi) \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|}, \ \langle \xi \rangle = (1+|\xi|^2)^{1/2}, \ \xi \in \mathbb{R}^n.$$

ΨDOs on the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. Fourier coefficients with $\xi \in \mathbb{Z}^n$,

$$\widehat{f}(\xi) = \int_{\mathbb{T}^n} f(x) e^{-2\pi i x\cdot\xi} dx, \quad T_\sigma f(x) = \sum_{\xi \in \mathbb{Z}^n} e^{2\pi i x\cdot\xi} \sigma(x,\xi) \widehat{f}(\xi)$$

$$\left| \Delta_\xi^\alpha \partial_x^\beta \sigma(x,\xi) \right| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\alpha|}, \ \xi \in \mathbb{Z}^n$$

[Agranovich 1990], [McLean 1991], [Turunen 2000], [R.+ Turunen, JFAA, 2010].

ΨDOs on a compact Lie group G:[R+Turunen, Birkhäuser book, 2010]

$$\widehat{f}(\xi) = \int_G f(x)\xi(x)^* dx,$$

$$T_\sigma f(x) = \sum_{[\xi]\in\widehat{G}} d_\xi \mathrm{Tr}\left(\xi(x)\sigma(x,\xi)\widehat{f}(\xi)\right),$$

$$||\Delta_\xi^\alpha X^\beta \sigma(x,\xi)||_{\mathrm{op}} \leq C_{\alpha\beta}\langle\xi\rangle^{m-|\alpha|}, \quad \xi \in \widehat{G},\ \langle\xi\rangle = \text{e.v.},\ \Delta_\xi = \text{diff. op.}, \ldots$$

Harmonic and Non-Harmonic Analysis

- Harmonic Analysis: symmetries in the underlying space, e.g. working with $e^{2\pi i x\cdot\xi}$ on $\mathbb{T}^n$ with $\xi \in \mathbb{Z}^n$;
  more generally, working with representations of compact, nilpotent, or more general locally compact type 1 groups;
- Non-Harmonic Analysis: no symmetries in the underlysing space, e.r. working with $e^{2\pi i x\cdot\xi}$ on $\mathbb{T}^n$ with $\xi \notin \mathbb{Z}^n$;
  Paley and Wiener (Fourier transforms in the complex domain, 1934) called this nonharmonic analysis;
  more generally, working with eigenfunction expansions for boundary value problems, or for compact and noncompact manifolds, with and without boundary;
  Nonharmonic Analysis of boundary value problems. R.+ Tokmagambetov, IMRN 2016; MMNP 2017;
  Compact manifolds with boundary: Delgado+R.+Tokmagambetov, JMPA, 2017.

Non-Harmonic Analysis Of BVPs

Setting: Let Ω be a smooth d-dimensional manifold with a boundary. Let L be a differential operator with smooth coefficients on Ω with boundary condition on $\partial\Omega$(or we can say that L has some domain).
Assumption: the spectrum of L is discrete: $Lu_\xi = \lambda_\xi u_\xi$, $\xi \in I$, and $\{u_\xi\}$ is a Riesz basis in $L^2(\Omega)$ (any element can be uniquely represented in this basis).
Note: L need not be self adjoint.

Adjoint problem: $L^* v_\xi = \overline{\lambda_\xi} v_\xi,\ \xi \in I$.
Bari(1951): u_ξ is a basis if and only if v_ξ is a basis.

Families $\{u_\xi\}$ and $\{v_\xi\}$ are biorthogonal: $(u_\xi, v_\eta)_{L^2(M)} = \delta_{\xi\eta}$.

Some examples

Classical Fourier analysis=decompositions with respect to eigenfunctions of $L = -i\frac{\partial}{\partial x}$, on $(0,1)$ with periodic boundary conditions $y(0) = y(1)$.
Indeed, this is a self-adjoint operator with an orthonormal basis given by $e^{2\pi i x\cdot\xi}$.

Let's change the above problem slightly.

- $\Omega = (0,1)$, $L = -i\frac{\partial}{\partial x}$, $hy(0) = y(1)$, $h > 0$.
  Titchmarsh 1926, Cartwright 1930: $\lambda_\xi = -i \ln h + 2\pi\xi, \quad \xi \in \mathbb{Z},$
  biorthogonal system $u_\xi(x) = h^x e^{2\pi i x\cdot\xi}$, $v_\xi(x) = h^{-x} e^{2\pi i x\cdot\xi}$
- orthogonal examples: harmonic oscillator, anharmonic oscillator, Landau Hamiltonian, Hörmander's sums of squares on compact manifolds, and many others.
  These can be made non-orthogonal by e.g adding some non-self-adjoint boundary conditions.

Global Fourier Analysis Associated to L and L^*

Recall: discrete spectrum $Lu_\xi = \lambda_\xi u_\xi$, $L^* v_\xi = \lambda_\xi v_\xi$, $\xi \in I$ discrete set.
$C_L^\infty(\Omega) = \cap_{k=1}^\infty \mathrm{Dom}(L^k)$, $C_{L^*}^\infty(\Omega) = \cap_{k=1}^\infty \mathrm{Dom}((L^*)^k)$.
$\mathcal{D}'_L(\Omega) = \mathcal{L}(C_{L^*}^\infty(\Omega), \mathbb{C})$, $\mathcal{D}'_{L^*}(\Omega) = \mathcal{L}(C_L^\infty(\Omega), \mathbb{C})$,

$$\mathcal{F}_L f(\xi) = \widehat{f}(\xi) := \int_M f(x) v_\xi(x) dx, \quad \mathcal{F}_{L^*} f(\xi) = \widehat{f}_*(\xi) := \int_M f(x) u_\xi(x) dx.$$

If L is a differential operator of order m on Ω, we define
$\langle \xi \rangle := (1 + |\lambda_\xi|)^{1/m}$.
$\mathcal{S}(I)$: space of $|h(\xi)| \leq C \langle \xi \rangle^{-M}$ for all M.

- $\mathcal{F}_L : C_L^\infty(\Omega) \to \mathcal{S}(I)$ and $\mathcal{F}_{L^*} : C_{L^*}^\infty(\Omega) \to \mathcal{S}(I)$ are bijective homeomorphism with the Fourier inversion formulae

$$\mathrm{f(x)} = \sum_{\xi \in I} \widehat{f}(\xi) u_\xi(x) = \sum_{\xi \in I} \widehat{f}_*(\xi) v_\xi(x).$$

- Extend to distributions, e.g. $\mathcal{F}_L : \mathcal{D}'_L(\Omega) \to S'(I)$

From the Riesz basis property, we have

$$m^2||f||_{L^2}^2 \leq \sum_{\xi \in I} |\widehat{f}(\xi)|^2 \leq M||f||_{L^2}^2.$$

- Plancheral Indentities:
  Define $(\mathrm{a,b})_{\ell^2_L} := \sum_{\xi \in I} a(\xi)(\mathcal{F}_{L^*} \circ \mathcal{F}_L^{-1} b)(\xi).$ Then
  $(f, g)_{L^2} = (\widehat{f}, \widehat{g})_{\ell^2_L} = \sum_{\xi \in I} \widehat{f}(\xi)\widehat{g}_*(\xi).$ Similarly with $\ell^2_{L^*}$, so that
  $||f||_{L^2} = ||\widehat{f}||_{\ell^2_L} = ||\widehat{f}_*||_{\ell^2_{L^*}}$
- Sobolev Space: Let $f \in \mathcal{D}'_L(\Omega) \cap \mathcal{D}'_{L^*}(\Omega)$ and $s \in \mathbb{R}$.
  $f \in H^s_L(\Omega)$ if $\langle \xi \rangle^s \widehat{f}(\xi) \in \ell^2_L$. It is a Hilbert space with a norm

$$||f||_{H^s_L(M)} := \left(\sum_{\xi \in I} \langle \xi \rangle^{2s} \widehat{f}(\xi) \widehat{f}_*(\xi) \right)^{1/2}$$

We can further define ℓ^p_L, $\ell^p_{L^*}$. These are interpolation spaces. Fourier transform satisfies Hausdroff- Young inequality and $(\ell^p_L)' = \ell^{p'}_{L^*}$. Here

$$||a||_{\ell^p_L} = \left(\sum_{\xi \in I} |a(\xi)|^p ||u_\xi||^{2-p}_{L^\infty} \right)^{1/p}, \quad \text{for } 1 \leq p \leq 2,$$

and

$$||a||_{\ell^p_L} = \left(\sum_{\xi \in I} |a(\xi)|^p ||v_\xi||^{2-p}_{L^\infty} \right)^{1/p}, \quad \text{for } 2 \leq p \leq \infty.$$

Difference operators

Next question: How to define symbol classes? need some operations in ξ.

A collection $q_j \in C^\infty(\Omega \times \Omega), j = 1, 2, ..., l$, of smooth functions on Ω is called $L-$strongly admissible if

- For every $x \in \Omega$, the multiplication by $q_j(x, .)$ is a continuous linear mapping on $C_L^\infty(\Omega)$, for all $j = 1, 2, ..., l$;
- $q_j(x, x) = 0$ for all $j = 1, 2, ..., l$;
- $\text{rank}(\nabla_y q_1(x, y), ..., \nabla_y q_l(x, y))|_{y=x} = \dim \Omega$;
- the diagonal in $\Omega \times \Omega$ is the only set when all of the q_j's vanish:
$$\cap_{j=1}^{l} \{(x, y) \in \Omega \times \Omega : q_j(x, y) = 0\} = \{(x, x) : x \in \Omega\}.$$

We will use the multi-index notation

$$q^\alpha(x, y) := q_1^{\alpha_1}(x, y)...q_l^{\alpha_1}(x, y).$$

Analogously, one defines L^*-strongly admissible collections.

Difference operators are not in general x-invariant

We define difference operator $\Delta^{\alpha}_{q,(x)}$ by any of the following equal expressions

$$\Delta^{\alpha}_{q,(x)}\sigma(x,\xi)(\xi) = u_{\xi}^{-1}(x)\int_{\Omega} q^{\alpha}(x,y)K(x,y)u_{\xi}(y)dy,$$

$K \in \mathcal{D}'_{L}(\Omega \times \Omega)$, Schwartz Kernel of the operator T_{σ}. Analogously, the difference operator $\widetilde{\Delta}^{\alpha}_{q,(x)}$ acting on adjoint Fourier coefficients by

$$\widetilde{\Delta}^{\alpha}_{\tilde{q},(x)}\sigma(x,\xi)(\xi) = v_{\xi}^{-1}(x)\int_{\Omega} \widetilde{q}^{\alpha}(x,y)\widetilde{K}(x,y)v_{\xi}(y)dy,$$

$K \in \mathcal{D}'_{L^*}(\Omega \times \Omega)$, Schwartz Kernel of the operator T_{σ}. The above definitions work if the eigenfunctions u_{ξ}, v_{ξ} do not have zeros. However, this assumption can be relaxed.(R.+ Tokmagambetov, MMNP 2017).
Difference operators with respect to ξ also depend on x.

Symbol Classes $S^m_{\rho,\delta}(\Omega)$

Global symbol classes $S^m_{1,0}(\Omega) = S^m(\Omega)$ consisting of functions $\sigma(x,\xi)$ which are smooth in x and satisfy

$$\left|\Delta^{\alpha}_{(x)} D^{(\beta)}_x \sigma(x,\xi)\right| \leq C\langle\xi\rangle^{m-|\alpha|}$$

Also class $S^m_{\rho,\delta}(\Omega)$ with

$$|\Delta^{\alpha}_{(x)} D^{(\beta)}_x \sigma(x,\xi)| \leq C\langle\xi\rangle^{m-\rho|\alpha|+\delta|\beta|}.$$

Some remarks:

- on $\mathbb{R}^n$, $\Delta^{\alpha}_{(x)} = \partial^{\alpha}_{\xi}$; on torus $\mathbb{T}^n$, these are difference operators Δ^{α}_{ξ} on $\mathbb{Z}^n \simeq \widehat{\mathbb{T}^n}$.
- for a Lie group G, difference operators were introduced and used on $\widehat{G}$ to define global Hörmander classes $S^m(G \times \widehat{G})$. There, $G\widehat{G}$ can be viewed as a global phase space.
- Here the difference operators $\Delta^{\alpha}_{(x)}$ in ξ are x-dependent!. This is somewhat natural since we do not have any underlying invariance.

Weighted Symbol Class

Weight Function: $\Lambda \in C_L^\infty(\mathcal{I})$ is a weight function if there exists suitable $\mu_0 \leq \mu_1 \leq \mu$ and C_0, C_1 such that for any multi-indices $\alpha, \gamma \geq 0$, $\gamma_j \in \{0,1\}, \forall j$, and $C_{\alpha,\gamma} > 0$

$$C_0 \langle \xi \rangle^{\mu_0} \leq \Lambda(\xi) \leq C_1 \langle \xi \rangle^{\mu_1}, \quad \xi \in \mathcal{I}$$

$$\left| \langle \xi \rangle^{|\gamma|} \Delta_{(x)}^{\alpha+\gamma} \Lambda(\xi) \right| \leq C_{\alpha,\gamma} \Lambda(\xi)^{1-(1/\mu)|\alpha|}, \quad \xi \in \mathcal{I}.$$

Symbol classes related to weight functions $S^m_{\rho,0,\Lambda}$, $\rho \in (0, 1/\mu]$ consisting of functions smooth in x and satisfy

$$\left| \Delta_{(x)}^{\alpha} D_x^{(\beta)} \sigma(x,\xi) \right| \leq C \Lambda(\xi)^{m-\rho|\alpha|}$$

- For $\Lambda(\xi) = (1 + |\lambda_\xi^2|)^{\frac{1}{2m}}$, $\xi \in \mathcal{I}$, $S^m_{\rho,0,\Lambda} =$ Hörmander class $S^m_{\rho,0}$, $m \in \mathbb{R}$ and $\rho \in (0,1]$.

Weighted M-Symbol Class (ADG+VK+LM+SSM)

Weighted M-symbol class $M^m_{\rho,0,\Lambda}$ to be the class of all such symbols which are smooth in x and satisfy

$$\langle\xi\rangle^{|\gamma|}\Delta^{\gamma}_{(x)}\sigma(x,\xi) \in S^m_{\rho,0,\Lambda},$$

for all γ such that $\gamma_j \in \{0,1\}$.

- For any $m \in \mathbb{R}$ and $0 < \rho \leq \frac{1}{\mu}$, there exist $N_0 > 0$, such that

$$S^{m-N_0}_{\rho,0,\Lambda} \subset M^m_{\rho,\Lambda} \subset S^m_{\rho,0,\Lambda}.$$

- The L-pseudo-differential operator is defined as

$$\mathrm{T}_\sigma f(x) = \sum_{\xi\in\mathcal{I}} \sigma(x,\xi)\widehat{f}(\xi)u_\xi(x),$$

for every $f \in C^\infty_L(\Omega)$.

L^p-boundedness, (ADG+VK+LM+SSM)

Theorem

For $\sigma \in M^0_{\rho,0,\Lambda}(\Omega \times \mathcal{I})$, the operator $Op_L(\sigma) : L^p(\Omega) \to L^p(\Omega)$ is a bounded operator.

- Weighted Sobolev Space: $H^{s,p}_{L,\Lambda} = \{w \in \mathcal{D}'(\Omega) : \Lambda(D)^s w \in L^p(\Omega)\}$. Norm, $||w||_{H^{s,p}_{L,\Lambda}} = ||\Lambda(D)^s w||_{L^p(\Omega)}$, and $H^{s,p}_{L,\Lambda}$ is a Banach space.

Theorem

For $\sigma \in M^m_{\rho,0,\Lambda}(\Omega \times \mathcal{I})$, the operator $Op_L(\sigma) : \mathcal{H}^{s,p}_{L,\Lambda} \to \mathcal{H}^{s-m,p}_{L,\Lambda}$ for any $s \in \mathbb{R}$ is a bounded operator.

Calculus

Theorem (Asymptotic sums of symbols, ADG+VK+LM+SSM)

Suppose that $\sigma_j \in M^{m_j}_{\rho,0,\Lambda}$ for all $j \in \mathbb{N}_0$, where $\{m_j\}_{j=0}^{\infty} \subset \mathbb{R}$ be a sequence such that $m_j > m_{j+1}$, and $m_j \to -\infty$ as $j \to \infty$. Then there exists a L-symbol $\sigma \in M^{m_0}_{\rho,0,\Lambda}$ such that for all $N \in \mathbb{N}_0$

$$\sigma \sim \sum_{j=0}^{N-1} \sigma_j.$$

Theorem (Adjoint, ADG+VK+LM+SSM)

Let $T : C_L^{\infty}(\Omega) \to C_L^{\infty}(\Omega)$ be a continuous linear operator such that its L-symbol $\sigma_T \in M^{m}_{\rho,0,\Lambda}$. Then the adjoint T^* of T is a L^*-pseudo-differntial operator with L^*-symbol $\sigma_{T^*} \in M^{m}_{\rho,0,\Lambda}$ having asymptotic expansion

$$\sigma_{T^*}(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \widetilde{\Delta}^{\alpha}_{(x)} D_x^{(\alpha)} \overline{\sigma_T(x,\xi)}$$

Theorem (Product, ADG+VK+LM+SSM)

Let $m_1, m_2 \in \mathbb{R}$. Let $A, B : C_L^\infty(\Omega) \to C_L^\infty(\Omega)$ be continuous linear operator such that $\sigma_A \in M_{\rho,0,\Lambda}^{m_1}$ and $\sigma_B \in M_{\rho,0,\Lambda}^{m_2}$. Then the symbol of AB, $\sigma_{AB} \in M_{\rho,0,\Lambda}^{m_1+m_2}$ having asymptotic expansion

$$\sigma_{AB}(x,\xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \left(\Delta_{(x)}^{\alpha} \sigma_A(x,\xi)\right) D_x^{(\alpha)} \sigma_B(x,\xi),$$

where the asymptotic expansion means that for every $N \in \mathbb{N}$, we have

$$\sigma_{AB}(x,\xi) - \sum_{|\alpha|<N} \frac{1}{\alpha!} \left(\Delta_{(x)}^{\alpha} \sigma_A(x,\xi)\right) D_x^{(\alpha)} \sigma_B(x,\xi) \in M_{\rho,0,\Lambda}^{m_1+m_2-\rho N}.$$

M-Elliptic Operators

Any $\sigma \in M^m_{\rho,0,\Lambda}$ is M-elliptic if there exists constant $C > 0$ and $R(> 0) \in \mathbb{R}$ such that

$$|\sigma(x,\xi)| \geq C(\Lambda(\xi))^m$$

for $|\lambda_\xi| \geq R$.

Theorem (ADG+VK+LM+SSM)

Let $A : C^\infty_L(\Omega) \to C^\infty_L(\Omega)$ continuous linear operator such that its L-symbol σ_A is M-elliptic. Then there exists a symbol $\sigma_B \in M^{-m}_{\rho,0,\Lambda}$ such that

$$BA = I + R$$

and

$$AB = I + S,$$

where the pseudo differential operators R, S are in $Op_L M^{-\infty}$.

Minimal and Maximal Operators

- $T_\sigma : L^2(\Omega) \to L^2(\Omega)$ is closable for $\sigma \in M^m_{\rho,0,\lambda}$, $m > 0$
- Maximal Operator: $g \in Dom(T_{\sigma,1})$ and $T_{\sigma,1} g = f$ if and only if

$$\langle g, T_\sigma^* \psi \rangle = \langle f, \psi \rangle,$$

where T_σ^* is the adjoint of T_σ and $\psi \in C^\infty(\bar{\Omega})$.

Results: ADG+VK+LM+SSM, arxiv 2023.

- For M-elliptic symbol $\sigma \in M^m_{\rho,0,\Lambda}$, $Dom(T_{\sigma,0}) = H^{m,2}_{L,\Lambda}$.
- $T_{\sigma,0} = T_{\sigma,1}$, for M-elliptic $\sigma \in M^m_{\rho,0,\Lambda}$, $m > 0$.
- Suppose $\sigma \in M^m_{\rho,0,\Lambda}$, $m > 0$ be M-elliptic and is independent of x. If $\lambda \in \mathbb{C}$ such that

$$\sigma(\xi) \neq \lambda,$$

then $\lambda \in \rho(T_{\sigma,0})$.

More Results (ADG+VK+LM+SSM)

Gohberg's lemma: Let $1 < p < \infty$. Assume Ω has a finite measure. Let $\sigma \in M^0_{\rho,0,\Lambda}$, $0 < \rho \leq 1$. Then for all compact operators $K \in \mathcal{L}(L^p(\Omega))$,

$$||T_\sigma - K||_{\mathcal{L}(L^p(\Omega))} \geq d_\sigma := \limsup_{\langle \xi \rangle \to \infty} \left\{ \sup_{x \in \Omega} |\sigma(x,\xi)| \right\}.$$

Compactness: Assume Ω has a finite measure. Let T_σ have symbol in $M^0_{\rho,0,\Lambda}$, $0 < \rho \leq 1$. Then T_σ extends to a compact operator in $L^2(\Omega)$, if and only if

$$\mathrm{d}_\sigma := \limsup_{\langle \xi \rangle \to \infty} \left\{ \sup_{x \in \Omega} |\sigma(x,\xi)| \right\} = 0.$$

Riesz Operator: Assume Ω has a finite measure. Let T_σ have symbol in $M^0_{\rho,0,\Lambda}$. The T_σ is a Riesz operator on $L^p(\Omega)$, $1 < p < \infty$ if and only if

$$\mathrm{d}_{\sigma'} := \lim_{\langle \xi \rangle \to \infty} \left\{ \sup_{x \in \Omega} |\sigma(x,\xi)| \right\} = 0.$$

More Results (ADG+VK+LM+SSM)

Functional Symbolic Calculus: Let $m > 0$, $0 < \rho \leq 1$ and $\sigma \in M^m_{\rho,0,\Lambda}$ be a L-elliptic , $\sigma > 0$. Then
$\widehat{B}(x,\xi) \equiv \sigma(x,\xi)^{\frac{1}{2}} := \exp\left(\frac{1}{2}\log(\sigma(x,\xi))\right) \in M^{\frac{m}{2}}_{\rho,0,\Lambda}$
Gärding's Inequality: Let $T_\sigma : C^\infty_L(\Omega) \to \mathcal{D}'_L(\Omega)$ with symbol $\sigma \in M^m_{\rho,0,\Lambda}$, $m > 0$ and $0 < \rho \leq 1$. Also assume

$$A(x,\xi) := \frac{1}{2}(\sigma(x,\xi) + \overline{\sigma(x,\xi)}), \qquad (x,\xi) \in \Omega \times \mathcal{I}$$

satisfies

$$|(\Lambda(\xi))^m A(x,\xi)^{-1}| \leq C_0$$

for some $C_0 > 0$. Then, there exists $C_1, C_2 > 0$ such that

$$\text{Re}(\sigma(x,D)u,u) \geq C_1||u||_{H^{\frac{m}{2},2}_{L,\Lambda}} - C_2||u||_{H^{0,2}_{L,\Lambda}}$$

holds true for every $u \in C^\infty_L(\Omega)$.

Applications

Theorem

Let $\sigma \in M^{2m}_{\rho,\Lambda}$, $m > 0$, be such that it satisfies the condition given in the Gärding's inequality. Then for all $f \in L^2(\Omega)$ there exists $\lambda_0 \in \mathbb{R}$, such that for all $\lambda \geq \lambda_0$,

$$(T_\sigma + \lambda I)u = f$$

on Ω has a unique strong solution $u \in L^2(\Omega)$.

Reference:

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Thank You!!